A Simplified Form of the Hermitianizing Matrix for the Half Integer Spin Bhabha Fields

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The hermitianizing matrix of the Bhabha fields for $s =$ half integer can be simplified to the form $f(x, s) = (-1)^{s-p-1/2}$ and as a result the charge of these fields for $s > 1/2$ is indefinite.

1. Introduction

Bhabha in his effort to free the higher spin theories from the presence of the subsidiary conditions, proposed an equation which is similar in appearance to the Dirac wave-equation and which in the absence of interactions reads

$$\mathbb{L}_0 \frac{\partial \psi}{\partial x_0} + \mathbb{L}_1 \frac{\partial \psi}{\partial x_1} + \mathbb{L}_2 \frac{\partial \psi}{\partial x_2} + \mathbb{L}_3 \frac{\partial \psi}{\partial x_3} + i \chi \psi = 0 , \quad (1)$$

where $\mathbb{L}_k$, $k = 0, 1, 2, 3$, are four matrices of appropriate dimension depending on the representation according to which the wave-function ψ transforms and χ is a constant related to the mass of the particle [1, 5].

The Bhabha field is a multimass and multispin field. (In a field of maximum spin s all the lower values of the spin appear as well.)

We shall limit ourselves here to those Bhabha wave-equations for which the underlying representation belongs to the group SO (4, 1). In this case the matrices $\mathbb{L}_k$ satisfy a relation of the form

$$[\mathbb{L}_m, \mathbb{L}_n]_- = \mathbb{I}_{mn} , \quad (2)$$

where $\mathbb{I}_{mn}$, $(m, n = 0, 1, 2, 3)$ are the infinitesimal generators of the Lorentz group. Furthermore a matrix $\mathbb{A}$ known as the hermitianizing matrix exists satisfying the following properties [6]:

$$\bar{\psi} = \psi^\dagger \mathbb{A}, \quad \mathbb{A}^2 = 1, \quad [\mathbb{L}_0, \mathbb{A}]_- = 0,$$
$$\{\mathbb{L}_\mu, \mathbb{A}\}_+ = 0, \qquad \mu = 1, 2, 3 \quad (3)$$

(where $\psi^\dagger$ is the hermitian conjugate of ψ).

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For the Bhabha wave-equations for half integer spin we prove that the hermitianizing matrix $\mathbb{A}$ can be written in a diagonal form with elements on the main diagonal given by the formula

$$f(x, s) = (-1)^{s-p-1/2} , \quad (4)$$

where s is the maximum value of the spin described by the field, x is a variable taking values s, $(s-1) \ldots -(s-1)$, $-s$, and p is another variable related to x by the formula $x = p + 1/2$. Finally, using this result, we show that the charge densities of the Bhabha fields for half integer spin $s \geqq 3/2$ are indefinite, which has as a consequence that the charge is indefinite.

2. A Simplified Form of $\mathbb{A}$

With every Bhabha wave equation one can associate the quantity ϱ_0 known as the charge density given by the formula

$$\varrho_0 = \psi^\dagger \mathbb{A} \mathbb{L}_0 \psi . \quad (5)$$

The total charge is given by

$$\varrho = \int \varrho_0 \, \mathrm{d}u \quad (6)$$

($\mathrm{d}u =$ volume element).

To be able to find ϱ_0 and hence the charge for any spin we need to know the eigenvalues of the product $\mathbb{A}\mathbb{L}_0$. An explicit expression of $\mathbb{L}_0$ is not necessary because in the case of the Bhabha field for any spin s the matrix $\mathbb{L}_0$ is diagonalizable and has eigenvalues $\pm s$, $\pm(s-1)$, $\pm(s-2) \ldots$ with multiplicities $\geqq 1$. An expression for $\mathbb{A}$ as a function of $\mathbb{L}_0$ for any spin s, which also gives the eigenvalues of $\mathbb{A}$ corresponding to the eigenvalues of $\mathbb{L}_0$ is given by

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Madava-Rao et al. [7] and is:

$$\mathbb{A}(s) = f(\mathbb{L}_0, s) \equiv f(x, s) = \frac{(x-s)(x-s+1)\dots(x+s-1)(x+s)}{(2s)!} \cdot \sum_{n=0}^{2s} \binom{2s}{n} \frac{1}{(x-s+n)}, \tag{7}$$

where s is the maximum value of the spin associated with the considered field and x any of the eigenvalues of $\mathbb{L}_0$. For s half integer, which is what we are concerned with in this paper the above formula reduces to

$$f(x, s) = \frac{2x(x^2-1/4)(x^2-9/4)\dots(x^2-s^2)}{(2s)!} \cdot \sum_{n=1}^{(s+1/2)} \binom{2s}{s+\frac{1}{2}-n} \frac{1}{[x^2-(n-\frac{1}{2})^2]}. \tag{8}$$

This formula, as shown below, can be simplified to the formula

$$f(x, s) = (-1)^{s-p-1/2}, \tag{9}$$

which can be used to calculate the eigenvalues of $\mathbb{A}$ corresponding to the eigenvalues of $\mathbb{L}_0$.

Proof:

Let us consider (8). In this formula for fixed x the number n takes the values $1, 2, 3, \dots (s-1/2)$, $(s+1/2)$. The variable x takes the values $\pm\frac{1}{2}$, $\pm\frac{3}{2}$, $\pm\frac{5}{2}, \dots \pm(s-1), \pm s$. In each summation (obtained for fixed x and n running from 1 to $s+1/2$) only one term survives. Finally all the non-zero contributions of the above formula when x takes values from $\pm\frac{1}{2}$ to $\pm s$, correspond to the following combinations of x and n:

$$\left(x=\pm\frac{1}{2}, n=1\right), \left(x=\pm\frac{3}{2}, n=2\right),$$

$$\left(x=\pm\frac{5}{2}, n=3\right)\dots\left(x=\pm\frac{q}{2}, n=\frac{q}{2}+\frac{1}{2}\right)\dots$$

$$\dots\left(x=\pm(s-1), n=\left(s-\frac{1}{2}\right)\right), \left(x=\pm s, n=s+\frac{1}{2}\right).$$

Let us now calculate the contribution corresponding to $\left(x=\pm\frac{q}{2}, n=\frac{q}{2}+\frac{1}{2}\right)$, where $\frac{q}{2}$ indicates an arbitrary choice of x in the interval $[-s, s]$. For $x=\frac{q}{2}$, $n=\frac{q}{2}+\frac{1}{2}$ the surviving term is

$$\frac{2x\left(x^2-\frac{1}{4}\right)\left(x^2-\frac{9}{4}\right)\dots\left(x^2-\left(\frac{q}{2}-2\right)^2\right)\left(x^2-\left(\frac{q}{2}-1\right)^2\right)\left(x^2-\left(\frac{q}{2}+1\right)^2\right)\left(x^2-\left(\frac{q}{2}+2\right)^2\right)\dots(x^2-(s-1)^2)(x^2-s^2)(2s)!}{(2s)!\left(s-\frac{q}{2}\right)!\left(s+\frac{q}{2}\right)!}$$

$$= \frac{2x(x-s)(x-s+1)\dots\left(x-\left(\frac{q}{2}+2\right)\right)\left(x-\left(\frac{q}{2}+1\right)\right)\left(x-\left(\frac{q}{2}-1\right)\right)\left(x-\left(\frac{q}{2}-2\right)\dots\left(x-\frac{3}{2}\right)\left(x-\frac{1}{2}\right)\right.}{1}$$

$$\cdot \frac{\left(x+\frac{1}{2}\right)\left(x+\frac{3}{2}\dots\left(x+\left(\frac{q}{2}-2\right)\right)\left(x+\left(\frac{q}{2}-1\right)\right)\left(x+\left(\frac{q}{2}+1\right)\right)\left(x+\left(\frac{q}{2}+2\right)\right)\dots(x+s-1)(x+s)\right.}{\left(s-\frac{q}{2}\right)!\left(s+\frac{q}{2}\right)!}$$

$$= \frac{2\frac{q}{2}\left(\frac{q}{2}-s\right)\left(\frac{q}{2}-s+1\right)\dots\left(\frac{q}{2}-\frac{q}{2}-2\right)\left(\frac{q}{2}-\frac{q}{2}-1\right)\left(\frac{q}{2}-\frac{q}{2}+1\right)\left(\frac{q}{2}-\frac{q}{2}+2\right)\dots\left(\frac{q}{2}-\frac{3}{2}\right)\left(\frac{q}{2}-\frac{1}{2}\right)}{1}$$

$$\cdot \frac{\left(\frac{q}{2}+\frac{1}{2}\right)\left(\frac{q}{2}+\frac{3}{2}\right)\dots\left(\frac{q}{2}+\frac{q}{2}-2\right)\left(\frac{q}{2}+\frac{q}{2}-1\right)\left(\frac{q}{2}+\frac{q}{2}+1\right)\left(\frac{q}{2}+\frac{q}{2}+2\right)\dots\left(\frac{q}{2}+s-1\right)\left(\frac{q}{2}+s\right)}{\left(s-\frac{q}{2}\right)!\left(s+\frac{q}{2}\right)!}$$

$$= \frac{q\left(\frac{q}{2}-s\right)\left(\frac{q}{2}-s+1\right)\dots(-2)(-1)(+1)(+2)\dots\left(\frac{q}{2}-\frac{3}{2}\right)\left(\frac{q}{2}-\frac{1}{2}\right)\left(\frac{q}{2}+\frac{1}{2}\right)\left(\frac{q}{2}+\frac{3}{2}\right)\dots}{1}$$

$$\cdot \frac{\dots (q-2)(q-1)(q+1)(q+2)\dots\left(\frac{q}{2}+s-1\right)\left(s+\frac{q}{2}\right)}{\left(s-\frac{q}{2}\right)!\left(s+\frac{q}{2}\right)!}$$

$$= \frac{q\left\{-\left(s-\frac{q}{2}\right)\right\}\left\{-\left(s-\frac{q}{2}-1\right)\right\}\dots\{-2\}\{-1\}(+1)(+2)\dots\left(\frac{q}{2}-\frac{3}{2}\right)\left(\frac{q}{2}-\frac{1}{2}\right)\left(\frac{q}{2}+\frac{1}{2}\right)\left(\frac{q}{2}+\frac{3}{2}\right)\dots}{1}$$

$$\frac{\dots (q-2)(q-1)(q+1)(q+2)\dots\left(\frac{q}{2}+s-1\right)\left(s+\frac{q}{2}\right)}{\left(s-\frac{q}{2}\right)!\left(s+\frac{q}{2}\right)!}$$

Shifting q from the front to the position between $(q-1)$ and $(q+1)$ and observing that the number of negative signs is $(s-q/2)$ (and hence a sign $(-1)^{s-q/2}$ comes out at the front) and also observing that the descending and ascending terms make factorials we find

$$f\left(x=\frac{q}{2}, s\right) = (-1)^{s-q/2}\frac{(s-q/2)!\,(s+q/2)!}{(s-q/2)!\,(s+q/2)!} = (-1)^{s-q/2}. \tag{10}$$

Similarly working we find for $x=-q/2$, $n=q/2+1/2$ that the surviving term has the value

$$f(x=-q/2, s) = (-1)^{s+q/2}. \tag{11}$$

Hence for any x with $x=\pm\frac{1}{2}, \pm\frac{3}{2}, \dots \pm s$ we find

$$f(x,s) = (-1)^{s-x}. \tag{12}$$

Setting $x=p+1/2$, where p is a number, we have

$$f(x,s) = (-1)^{s-p-1/2}. \tag{13}$$

3. Applications

As applications of this we make use of it to the study of the charge of the Bhabha fields for $s=\frac{1}{2}, \frac{3}{2}$ and in general for any half integer value of the spin.

i) Dirac field $(s=1/2)$

The eigenvalues of the matrix $\mathbb{L}_0$ in the case of the Dirac field are $x=\pm\frac{1}{2}$ (both double). Using (9) we find that the hermitianizing matrix has the following eigenvalues:

for $x=\frac{1}{2}$ and

$p=0, f(x=\frac{1}{2}, s=\frac{1}{2})=(-1)^{1/2-1/2}=1$ (double),

for $x=-\frac{1}{2}$ and

$p=-1, f(x=-\frac{1}{2}, s=\frac{1}{2})=(-1)^{1/2+1-1/2}=-1$ (double).

Then the eigenvalues $\Lambda_j = x\,f(x,s)$ of $\mathbb{A}\mathbb{L}_0$ are respectively

$$\Lambda_1=\Lambda_2=\tfrac{1}{2}, \quad \Lambda_3=\Lambda_4=\tfrac{1}{2},$$

and the charge density is

$$\varrho_0 = \tfrac{1}{2}\{\psi_1^*\psi_1+\psi_2^*\psi_2+\psi_3^*\psi_3+\psi_4^*\psi_4\},$$

which gives a definite charge for the Dirac equation. (Notice that the factor 1/2 is due to the choice of the eigenvalues of $\mathbb{L}_0$ as $\pm 1/2$ rather than ± 1 as normally chosen.)

ii) Spin $\frac{3}{2}$ Bhabha field

The eigenvalues of $\mathbb{L}_0$ are $x=\pm\frac{1}{2}, \pm\frac{3}{2}$ and the eigenvalues of the hermitianizing matrix are:

for $x=\frac{1}{2}$ and

$p=0, \quad f(x=\frac{1}{2}, s=\frac{3}{2})=(-1)^{3/2-1/2}=-1,$

for $x=-\frac{1}{2}$ and

$p=-1, f(x=-\frac{1}{2}, s=\frac{3}{2})=(-1)^{3/2+1-1/2}=1,$

for $x=\frac{3}{2}$ and

$p=1, \quad f(x=\frac{3}{2}, s=\frac{3}{2})=(-1)^{3/2-1-1/2}=1,$

for $x=-\frac{3}{2}$ and

$p=-2, f(x=-\frac{3}{2}, s=\frac{3}{2})=(-1)^{3/2+2-1/2}=-1.$

Hence the eigenvalues of $\mathbb{A}\mathbb{L}_0$ are respectively

$$\Lambda_1=\Lambda_2=-\tfrac{1}{2}, \quad \Lambda_3=\Lambda_4=\tfrac{3}{2}$$

and the charge density is

$$\varrho_0 = -\tfrac{1}{2}\psi_1^*\psi_1-\tfrac{1}{2}\psi_2^*\psi_2+\tfrac{3}{2}\psi_3^*\psi_3+\tfrac{3}{2}\psi_4^*\psi_4+\dots,$$

which gives an indefinite charge for the spin 3/2 Bhabha field.

iii) General case, s = half integer > 1/2

The eigenvalues of $\mathbb{L}_0$ in the general case are $\pm s$, $\pm (s-1)$, $\pm (s-2) \ldots$ with multiplicities $\geqq 1$. Using (9) we find that the hermitianizing matrix $\mathbb{A}$ has the eigenvalues:

for $x = s$ and

$p = s - \frac{1}{2}$, $f(x = s, s) = (-1)^{s-s+1/2-1/2} = 1$,

for $x = -s$ and

$p = -s - \frac{1}{2}$,

$f(x = -s, s) = (-1)^{s+s+1/2-1/2} = (-1)^{2s} = -1$

(since s is half integer and $2s$ is then an odd number),

for $x = s - 1$ and $p = s - 1 - \frac{1}{2}$,

$f(x = s-1, s) = (-1)^{s-s+1+1/2-1/2} = -1$,

for $x = -s + 1$ and $p = -s + 1 - \frac{1}{2}$,

$f(x = -s+1, s) = (-1)^{s+s-1+1/2-1/2} = (-1)^{2s-1} = 1$,

(since $2s - 1$ is even)

for $x = s - 2$ and $p = s - 2 - \frac{1}{2}$,

$f(x = s-2, s) = (-1)^{s-s+2+1/2-1/2} = 1$,

for $x = -s + 2$ and $p = -s + 2 - \frac{1}{2}$,

$f(x = -s+2, s) = (-1)^{s+s-2+1/2-1/2} = (-1)^{2s-2} = -1$

(since $2s - 2$ is odd), and so on.

The corresponding eigenvalues Λ_i of $\mathbb{A}\mathbb{L}_0$ are

$$\Lambda_1 = \Lambda_2 = s\,, \quad \Lambda_3 = \Lambda_4 = -(s-1)\,,$$
$$\Lambda_5 = \Lambda_6 = (s-2), \ldots,$$

and the charge density is

$$\varrho_0 = \psi^\dagger \mathbb{A}\mathbb{L}_0 \psi = s\,\psi_1^* \psi_1 + s\,\psi_2^* \psi_2 - (s-1)\,\psi_3^* \psi_3 - (s-1)\,\psi_4^* \psi_4 + (s-2)\,\psi_5^* \psi_5 + (s-2)\,\psi_6^* \psi_6 + \ldots$$

which gives an indefinite charge.

4. Summary

Having shown that the hermitianizing matrix of the Bhabha field for half integer spin can be simplified to the form given by (9) we were able to verify that the charge associated with these fields for $s > 1/2$ is indefinite.

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